

An isomorphic version of the slicing problem

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Abstract

Here we show that any centrally symmetric convex body $K \subset \mathbb{R}^n$ has a perturbation $T \subset \mathbb{R}^n$ which is convex and centrally symmetric, such that the isotropic constant of T is universally bounded. T is close to K in the sense that the Banach-Mazur distance between T and K is $O(\log n)$. If K has a non-trivial type then the distance is universally bounded. In addition, if $K \subset \mathbb{R}^n$ is quasi-convex then there exists a quasi-convex $T \subset \mathbb{R}^n$ with a universally bounded isotropic constant and with a universally bounded distance to K .

1 Introduction

Let $K \subset \mathbb{R}^n$ be a centrally symmetric (i.e. $K = -K$) convex set with a non empty interior. Such sets are referred to here as “bodies”. We denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ the standard scalar product and Euclidean norm in \mathbb{R}^n . We also write D for the unit Euclidean ball and $S^{n-1} = \partial D$. The body K has a unique linear image \tilde{K} with $\text{Vol}(\tilde{K}) = 1$ such that

$$\int_{\tilde{K}} \langle x, \theta \rangle^2 dx \quad (1)$$

does not depend on the choice of $\theta \in S^{n-1}$. We say that \tilde{K} is an isotropic linear image of K or that \tilde{K} is in isotropic position. The quantity in (1), for an arbitrary $\theta \in S^{n-1}$, is usually referred to as L_K^2 or as the square of the isotropic constant of K . An equivalent definition of L_K which does not involve linear images is the following:

$$nL_K^2 = \inf_T \int_K |Tx|^2 dx \quad (2)$$

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where the infimum is over all matrices T such that $\det(T) = 1$. For a comprehensive discussion of the isotropic position and the isotropic constant we refer the reader to [MP].

L_K is an important linearly invariant parameter associated with K . A major conjecture is whether there exists a universal constant $c > 0$ such that $L_K < c$ for all convex centrally symmetric bodies in all dimensions. A proof of this conjecture will have various consequences. Among others (see [MP]) it will establish the fact that any body of volume one has at least one $n - 1$ dimensional section whose volume is greater than some positive universal constant. This conjecture is known as the slicing problem or the hyperplane conjecture. The best estimate known to date is $L_K < cn^{1/4} \log n$ for $K \subset \mathbb{R}^n$ and is due to Bourgain [Bou2] (see also the presentation in [D]). In addition, for large classes of bodies the conjecture was positively verified (some examples of references are [Ba2], [Bou1], [KMP], [MP]).

In this note we deal with a known relaxation of this conjecture, which we call the “isomorphic slicing problem”. It was suggested to the author by V. Milman. For two sets $K, T \subset \mathbb{R}^n$, we define their “geometric distance” as

$$d_G(K, T) = \inf \left\{ ab; \frac{1}{a}K \subset T \subset bK, a, b > 0 \right\}.$$

The Banach-Mazur distance of K and T is

$$d_{BM}(K, T) = \inf \{ d_G(K, L(T)) ; L \text{ is a linear operator} \}.$$

Let $K_n, T_n \subset \mathbb{R}^n$ for $n = 1, 2, \dots$ be a sequence of bodies such that $d_{BM}(K, T) < Const$ independent of the dimension n . In this case we say that the families $\{K_n\}$ and $\{T_n\}$ are uniformly isomorphic. Indeed, the norms that K_n and T_n are their unit balls are uniformly isomorphic. The isomorphic slicing problem asks whether the slicing problem is correct, at least up to a uniform isomorphism. Formally:

Question 1.1 *Does there exist constants $c_1, c_2 > 0$ such that for any dimension n , for any body $K \subset \mathbb{R}^n$, there exists a body $T \subset \mathbb{R}^n$ with $d_{BM}(K, T) < c_1$ and $L_T < c_2$?*

In this note we answer this question affirmatively, up to a logarithmic factor. The following is proved here:

Theorem 1.2 *For any centrally symmetric convex body $K \subset \mathbb{R}^n$ there exists a centrally symmetric convex body $T \subset \mathbb{R}^n$ with $d_{BM}(K, T) < c_1 \log n$ and*

$$L_T < c_2$$

where $c_1, c_2 > 0$ are numerical constants.

The $\log n$ factor in Theorem 1.2 comes from the use of the l -position and Pisier's estimate for the norm of the Rademacher projection (see [P]). Actually we prove, in the notation of Theorem 1.2, that $d_{BM}(K, T) < c_1 M(K) M^*(K)$ (see definitions in Section 3). Therefore we verify the validity of the isomorphic slicing conjecture for bodies that have a linear image with bounded MM^* . This is a large class of bodies, including all bodies with a non trivial type. In addition, Proposition 5.2 and Proposition 5.3 provide other classes of bodies for which Question 1.1 has a positive answer.

There exist some connections between the slicing problem and its isomorphic versions. An example is provided in the following lemma.

Lemma 1.3 *Assume that there exist $c_1, c_2 > 0$ such that for any integer n and an isotropic body $K \subset \mathbb{R}^n$ there exists an isotropic body $T \subset \mathbb{R}^n$ with $d_G(K, T) < c_1$ and $L_T < c_2$. Then there exists $c_3 > 0$ such that for any integer n and a body $K \subset \mathbb{R}^n$, we have $L_K < c_3$.*

Proof: $L_T < c_2$, therefore T is in M -position (as observed by K. Ball, see definitions and proofs in [MP]). Since $d_G(K, T) < c_1$, also K is in M -position. Using Proposition 1.4 in [BKM] we obtain a universal bound for the isotropic constant. \square

A set $K \subset \mathbb{R}^n$ is quasi-convex with constant $C > 0$ if $\text{conv}(K) \subset CK$, where conv denotes convex hull. For centrally symmetric quasi-convex sets, the isomorphic slicing problem has a positive answer. Formally, as is proved in Section 4,

Theorem 1.4 *For any $C > 1$, there exist $c_1, c_2 > 0$ with the following property: If $K \subset \mathbb{R}^n$ is centrally symmetric and quasi-convex with constant C , then there exists a centrally symmetric $T \subset \mathbb{R}^n$ such that $d_{BM}(K, T) < c_1$ and $L_T < c_2$. (Note that necessarily T is $c_1 C$ -quasi-convex).*

Our proof has few consequences which are formulated and proved in Section 5, among which is an improvement of an estimate from [BKM]. Throughout this paper the letters $c, C, c', c_1, c_2, \text{Const}$ etc. denote some positive numerical constants, whose value may differ in various appearances. We ignore measurability issues as they are non-essential to our discussion. All sets and functions used here are assumed to be measurable.

2 Log concave functions

In this section we collect a few facts regarding log-concave functions, most of which are known and appear in [Ba1] or [MP], yet our versions

are slightly different. $f : \mathbb{R}^n \rightarrow [0, \infty)$ is log-concave if $\log f$ is concave on its support. f is s -concave, for $s > 0$, if $f^{1/s}$ is concave on its support. Any s -concave function is also log-concave (see e.g. [Bo]). Given a non-negative function f on \mathbb{R}^n we define for $x \in \mathbb{R}^n$,

$$\|x\|_f = \left(\int_0^\infty f(rx) r^{n+1} dr \right)^{-1/n+2}.$$

We also define $K_f = \{x \in \mathbb{R}^n; \|x\|_f \leq 1\}$. The following Busemann-type theorem appears in [Ba1] (see also [MP]):

Theorem 2.1 *Let f be an even log-concave function on \mathbb{R}^n . Then K_f is convex and centrally symmetric and $\|\cdot\|_f$ is a norm.*

In the sequel we make a repeated use of the following well known facts. The first is that for any $1 \leq k \leq n$,

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} < \left(e\frac{n}{k}\right)^k. \quad (3)$$

The second is that for any integers $a, b \geq 0$,

$$\int_0^1 s^a (1-s)^b ds = \frac{1}{(a+b+1) \binom{a+b}{a}}. \quad (4)$$

Lemma 2.2 *Let $f : \mathbb{R}^n \rightarrow [0, \infty)$ be an even function whose restriction to any straight line through the origin is s -concave. Assume that $f(0) = 1$. If $s > n$ then*

$$d_G(K_f, \text{Supp}(f)) < c \frac{s}{n}$$

where $c > 0$ is a numerical constant, and $\text{Supp}(f) = \{x; f(x) > 0\}$.

Proof: Fix $\theta \in S^{n-1}$. Denote $M_\theta = \sup\{r > 0; f(r\theta) > 0\}$. Since $f|_{\theta\mathbb{R}}$ is s -concave and $f(0) = 1$, for all $0 \leq r \leq M_\theta$,

$$f(r\theta) \geq \left(1 - \frac{r}{M_\theta}\right)^s.$$

By the definition of $\|\theta\|_f$ and (4),

$$\|\theta\|_f^{-(n+2)} \geq \int_0^{M_\theta} \left(1 - \frac{r}{M_\theta}\right)^s r^{n+1} dr = \frac{M_\theta^{n+2}}{(n+s+2) \binom{n+s+1}{n+1}}.$$

In addition, since $f|_{\theta\mathbb{R}}$ is even, its maximum is $f(0) = 1$ and

$$\|\theta\|_f^{-(n+2)} \leq \int_0^{M_\theta} r^{n+1} dr = \frac{1}{n+2} M_\theta^{n+2}.$$

Combining with the estimate (3),

$$\frac{(n+2)^{1/(n+2)}}{M_\theta} \leq \|\theta\|_f \leq \frac{e(n+s+2)^{1/n+2} \left(\frac{n+s+1}{n+1}\right)^{\frac{n+1}{n+2}}}{M_\theta}$$

and because $s > n$,

$$\forall \theta \in S^{n-1}, \frac{c_1}{M_\theta} < \|\theta\|_f < \frac{c_2}{M_\theta} \frac{s}{n} \Rightarrow \frac{n}{c_2 s} \text{Supp}(f) \subset K_f \subset \frac{1}{c_1} \text{Supp}(f)$$

and the lemma is proved. \square

The isotropic constant and the isotropic position may be defined for arbitrary measures or densities, rather than just for convex bodies. Let $f : \mathbb{R}^n \rightarrow [0, \infty)$ be an even function with $\int_{\mathbb{R}^n} f < \infty$. The entries of its covariance matrix with respect to a fixed orthonormal basis $\{e_1, \dots, e_n\}$ are defined as

$$M_{i,j} = \frac{1}{\int_{\mathbb{R}^n} f(x) dx} \int_{\mathbb{R}^n} f(x) \langle x, e_i \rangle \langle x, e_j \rangle dx.$$

We define $L_f = \left(\frac{f(0)}{\int_{\mathbb{R}^n} f}\right)^{\frac{1}{n}} \det(M)^{\frac{1}{2n}}$. One can verify that if f is the characteristic function of a body $K \subset \mathbb{R}^n$, then $L_f = L_K$. Our next lemma claims that if f is log-concave, the body K_f shares the isotropic constant of the function f , up to a universal constant. This fact appears in [MP] and in [Ba1], but not in a very explicit formulation. For completeness we present a proof here.

Lemma 2.3 *Let f be an even function on \mathbb{R}^n whose restriction to any straight line through the origin is log-concave. Assume that $\int_{\mathbb{R}^n} f < \infty$. Then,*

$$c_1 L_f < L_{K_f} < c_2 L_f$$

where $c_1, c_2 > 0$ are universal constants.

Proof: Multiplying f by a constant if necessary, we may assume that $f(0) = 1$. Integrating in polar coordinates, for any $y \in \mathbb{R}^n$,

$$\begin{aligned} & \int_{K_f} \langle x, y \rangle^2 dx \\ &= \int_{S^{n-1}} \int_0^{1/\|\theta\|_f} \langle y, r\theta \rangle^2 r^{n-1} dr d\theta = \frac{1}{n+2} \int_{S^{n-1}} \langle y, \theta \rangle^2 \frac{1}{\|\theta\|_f^{n+2}} d\theta \\ &= \frac{1}{n+2} \int_0^\infty \int_{S^{n-1}} f(r\theta) \langle y, \theta \rangle^2 r^{n+1} dr d\theta = \frac{1}{n+2} \int_{\mathbb{R}^n} \langle x, y \rangle^2 f(x) dx \end{aligned}$$

where $d\theta$ is the induced surface area measure on S^{n-1} . Denote by $M(f)$ and $M(K_f)$ the inertia matrices of f and the characteristic function of K_f , correspondingly. Then $Vol(K_f)M(K_f) = \frac{1}{n+2} \left(\int_{\mathbb{R}^n} f \right) M(f)$. To compare the isotropic constants, we need to estimate $\frac{\int f}{Vol(K_f)}$. Now,

$$Vol(K_f) = \frac{1}{n} \int_{S^{n-1}} \left(\int_0^\infty f(r\theta) r^{n+1} dr \right)^{\frac{n}{n+2}} d\theta. \quad (5)$$

We shall use the following one dimensional lemma, to be proved in the end of this section (see also [Ba1], [BKM] or [MP]).

Lemma 2.4 *Let $g : [0, \infty) \rightarrow [0, \infty)$ be a non-increasing log-concave function with $g(0) = 1$ and $\int_0^\infty g(t)t^{n-1}dt < \infty$. Then, for any integer $n \geq 1$,*

$$\frac{n^{\frac{n+2}{n}}}{n+2} \leq \frac{\int_0^\infty g(t)t^{n+1}dt}{\left(\int_0^\infty g(t)t^{n-1}dt\right)^{\frac{n+2}{n}}} \leq \frac{(n+1)!}{((n-1)!)^{\frac{n+2}{n}}}.$$

(the left most inequality - which is more important for us - is true without the log-concavity assumption).

Since f is even and log-concave on any line through the origin, it is non-increasing on any ray that starts in the origin. From the left most inequality in Lemma 2.4, for any $\theta \in S^{n-1}$ (except for a set of measure zero where the integral diverges),

$$\int_0^\infty f(r\theta) r^{n+1} dr \geq \frac{n^{\frac{n+2}{n}}}{n+2} \left(\int_0^\infty f(r\theta) r^{n-1} dr \right)^{\frac{n+2}{n}}$$

and according to (5),

$$Vol(K_f) \geq \frac{1}{n} \frac{n^{\frac{n+2}{n}}}{n+2} \int_{S^{n-1}} \int_0^\infty f(r\theta) r^{n-1} dr d\theta = \frac{n^{2/n}}{n+2} \int_{\mathbb{R}^n} f.$$

Since $M(K_f) = \frac{1}{n+2} \frac{\int_{\mathbb{R}^n} f}{Vol(K_f)} M(f)$,

$$\frac{L_{K_f}^2}{L_f^2} = \frac{1}{n+2} \left(\frac{\int_{\mathbb{R}^n} f}{Vol(K_f)} \right)^{1+\frac{2}{n}} \leq \frac{1}{n+2} \left(\frac{n+2}{n^{2/n}} \right)^{\frac{n+2}{n}} < c_2.$$

This finishes the proof of one part of the lemma. The proof of the other inequality is similar. Using the right most inequality in Lemma 2.4,

$$\frac{L_{K_f}^2}{L_f^2} = \frac{1}{n+2} \left(\frac{\int_{\mathbb{R}^n} f}{Vol(K_f)} \right)^{1+\frac{2}{n}} \geq \frac{1}{n+2} \left(\frac{n((n-1)!)^{\frac{n+2}{n}}}{(n+1)!} \right)^{\frac{n+2}{n}} > c_1$$

and the lemma is proved. \square

Proof of Lemma 2.4: Start with the left-most inequality. Define $A > 0$ such that $\int_0^\infty g(t)t^{n-1}dt = \int_0^A t^{n-1}dt$. Then,

$$\begin{aligned} & \int_0^A (1-g(t))t^{n+1}dt - \int_A^\infty g(t)t^{n+1}dt \\ & \leq A^2 \left[\int_0^A (1-g(t))t^{n-1}dt - \int_A^\infty g(t)t^{n-1}dt \right] = 0. \end{aligned}$$

Since $\int_0^A t^{n+1}dt = \frac{n+2}{n+2} \left(\int_0^A t^{n-1}dt \right)^{\frac{n+2}{n}}$, we get that

$$\int_0^\infty g(t)t^{n+1}dt \geq \int_0^A t^{n+1}dt = \frac{n+2}{n+2} \left(\int_0^\infty g(t)t^{n-1}dt \right)^{\frac{n+2}{n}}.$$

To obtain the other inequality we need to use the log-concavity of the function. Define $B > 0$ such that $h(t) = e^{-Bt}$ satisfies

$$\int_0^\infty g(t)t^{n-1}dt = \int_0^\infty h(t)t^{n-1}dt.$$

It is impossible that always $g < h$ or always $g > h$, hence necessarily $t_0 = \inf\{t > 0; h(t) \geq g(t)\}$ is finite. $-\log g$ is convex and vanishes at zero, so $\tilde{g}(t) = \frac{-\log g(t)}{t}$ is non-decreasing. Hence $(B - \tilde{g}(t))(t - t_0) \geq 0$ or equivalently $(h(t) - g(t))(t - t_0) \geq 0$ for all $t > 0$. Therefore,

$$\begin{aligned} & \int_0^{t_0} (g(t) - h(t))t^{n+1}dt - \int_{t_0}^\infty (h(t) - g(t))t^{n+1}dt \\ & \leq t_0^2 \left[\int_0^{t_0} (g(t) - h(t))t^{n-1}dt - \int_{t_0}^\infty (h(t) - g(t))t^{n-1}dt \right] = 0. \end{aligned}$$

Since $\int_0^\infty e^{-tB}t^{n+1}dt = \frac{(n+1)!}{((n-1)!)^{\frac{n+2}{n}}} \left(\int_0^\infty e^{-tB}t^{n-1}dt \right)^{\frac{n+2}{n}}$,

$$\int_0^\infty g(t)t^{n+1}dt \leq \int_0^\infty h(t)t^{n+1}dt = \frac{(n+1)!}{((n-1)!)^{\frac{n+2}{n}}} \left(\int_0^\infty g(t)t^{n-1}dt \right)^{\frac{n+2}{n}}.$$

\square

Denote $L_n = \sup_{K \subset \mathbb{R}^n} L_K$, where the supremum is over all bodies in \mathbb{R}^n . Define also $\tilde{L}_n = \sup_{f: \mathbb{R}^n \rightarrow [0, \infty)} L_f$ where the supremum is over all log-concave even functions on \mathbb{R}^n . Apriori, $L_n \leq \tilde{L}_n$ since characteristic functions of bodies are log-concave. Using the fact that convolution of log-concave functions is again log-concave, and convolving the characteristic function of K with itself, we even obtain that

$\sqrt{2}L_n \leq \tilde{L}_n$. Less trivial is the fact that the opposite inequality also holds, up to some constant (I was informed that Corollary 2.5 also appears in K. Ball's PhD thesis).

Corollary 2.5 *There exists $c > 0$ such that for any integer n ,*

$$\tilde{L}_n < cL_n.$$

Proof: By lemma 2.3, for any log-concave even $f : \mathbb{R}^n \rightarrow [0, \infty)$,

$$L_f < cL_{K_f}$$

and hence $\tilde{L}_n \leq cL_n$. □

3 Building a function on K

Let $K \subset \mathbb{R}^n$ be a body. In this section we find an αn -concave function F supported on K whose isotropic constant is bounded. From Lemma 2.3 it follows that $L_{K_F} < \text{Const}$. According to Lemma 2.1, K_F is a convex body, and by Lemma 2.2 we get that $d_G(K, K_F) < c\alpha$. By obtaining good estimates on α Theorem 1.2 would follow. Let $\|\cdot\|$ be the norm that K is its unit ball, and denote by σ the unique rotation invariant probability measure on S^{n-1} . The median of $\|x\|$ on S^{n-1} with respect to σ is referred to as $M'(K)$. We abbreviate $M' = M'(K)$ and define the following function on K :

$$f_K(x) = \inf \left\{ 0 \leq t \leq 1; x \in (1-t) \left[K \cap \frac{1}{M'} D \right] + tK \right\}.$$

Then f_K is a convex function which is zero on $K \cap \frac{1}{M'} D$. Define also

$$M(K) = \int_{S^{n-1}} \|x\| d\sigma(x), \quad M^*(K) = \int_{S^{n-1}} \|x\|_* d\sigma(x)$$

where $\|x\|_* = \sup_{y \in K} \langle x, y \rangle$ is the dual norm. It is known (e.g. [MS]) that $M(K)$ is comparable to $M'(K)$.

Proposition 3.1 *Let $K \subset \mathbb{R}^n$ be a body, and let $\alpha = cM(K)M^*(K)$. Then,*

$$\int_K (1 - f_K(x))^{\alpha n} dx < 2 \text{Vol} \left(K \cap \frac{1}{M'} D \right)$$

where $c > 0$ is some numerical constant.

Proof: We denote $F(x) = (1 - f(x))^{\alpha n}$. Then,

$$\begin{aligned}\int_K F(x)dx &= \int_0^1 \text{Vol}\{x \in K; F(x) \geq t\}dt \\ &= \int_0^1 \text{Vol}\{x \in K; f(x) \leq 1 - t^{\frac{1}{\alpha n}}\}dt\end{aligned}$$

and changing variables $s = 1 - t^{\frac{1}{\alpha n}}$ yields

$$\int_K F(x)dx = \alpha n \int_0^1 (1-s)^{\alpha n-1} \text{Vol}\left((1-s)\left[K \cap \frac{1}{M'}D\right] + sK\right) ds.$$

Expand the volume term into mixed volumes (see e.g. [Sch]):

$$\text{Vol}\left((1-s)\left[K \cap \frac{1}{M'}D\right] + sK\right) = \sum_{i=0}^n \binom{n}{i} V_i s^i (1-s)^{n-i}$$

where $V_i = V(K, i; [K \cap \frac{1}{M'}D], n-i)$. Therefore,

$$\int_K F(x)dx = \alpha n \sum_{i=0}^n V_i \binom{n}{i} \int_0^1 s^i (1-s)^{(\alpha+1)n-i-1} ds$$

and by (4),

$$\int_K F(x)dx = \frac{\alpha}{\alpha+1} V_0 \sum_{i=0}^n \frac{\binom{n}{i}}{\binom{(1+\alpha)n-1}{i}} \frac{V_i}{V_0}.$$

Using (3) we may write

$$\int_K F(x)dx = \frac{\alpha}{\alpha+1} V_0 \left[1 + \sum_{i=1}^n \left(c_{n,i} \frac{n}{(1+\alpha)n-1} \left(\frac{V_i}{V_0} \right)^{1/i} \right)^i \right] \quad (6)$$

where $\frac{1}{e} \leq c_{n,i} \leq e$. By Alexandrov-Fenchel inequalities $V_i^2 \geq V_{i-1} V_{i+1}$ for $i \geq 1$ (e.g. [Sch]). It follows that for $1 \leq i \leq j$,

$$\left(\frac{V_i}{V_0} \right)^{1/i} \geq \left(\frac{V_j}{V_0} \right)^{1/j}. \quad (7)$$

In particular, if $\alpha+1 > 4e \frac{V_1}{V_0}$, then by (7),

$$c_{n,i} \frac{n}{(1+\alpha)n-1} \left(\frac{V_i}{V_0} \right)^{1/i} < \frac{2e}{1+\alpha} \frac{V_1}{V_0} \leq \frac{1}{2}.$$

Substituting in (6) we obtain

$$\int_K F(x)dx < V_0 \sum_{i=0}^n \frac{1}{2^i} < 2V_0 = 2Vol\left(K \cap \frac{1}{M'}D\right).$$

It remains only to show that our $\alpha = cM(K)M^*(K)$ is greater than a constant times $\frac{V_1}{V_0}$. Since $\frac{1}{M'}D \cap K \subset \frac{1}{M'}D$,

$$\begin{aligned} V_1 &= V(K, 1; \left[K \cap \frac{1}{M'}D\right], n-1) \\ &\leq V\left(K, 1; \frac{1}{M'}D, n-1\right) = \frac{1}{(M')^{n-1}}Vol(D)M^*(K) \end{aligned}$$

because $Vol(D)M^*(K) = V(K, 1; D, n-1)$ (see e.g. [Sch]). Regarding V_0 , since M' is the median,

$$\sigma(M'K \cap S^{n-1}) \geq \frac{1}{2} \quad \Rightarrow \quad Vol\left(K \cap \frac{1}{M'}D\right) \geq \frac{Vol\left(\frac{1}{M'}D\right)}{2}.$$

To conclude,

$$\frac{V_1}{V_0} \leq \frac{1}{(M')^{n-1}}Vol(D)M^*(K) \frac{2}{\frac{1}{(M')^n}Vol(D)} = 2M'(K)M^*(K).$$

The median of a positive function is not larger than twice its expectation, therefore $M'(K) \leq 2M(K)$, and we get that for $\alpha = cM(K)M^*(K)$, it is true that $\alpha + 1 > 4e\frac{V_1}{V_0}$ for a suitable numerical constant $c > 0$. \square

Corollary 3.2 *Let $K \subset \mathbb{R}^n$ be a body, $\alpha = cM(K)M^*(K)$ and denote $F(x) = (1 - f_K(x))^{\alpha n}$. Then,*

$$L_F < c'$$

where c is the constant from Proposition 3.1 and $c' > 0$ is a numerical constant.

Proof: Consider F as a density on K , i.e. consider the probability measure $\mu_F(A) = \frac{\int_A F(x)dx}{\int_K F(x)dx}$. Since $F \equiv 1$ on $K \cap \frac{1}{M'}D$, by Proposition 3.1,

$$\mu\left(K \cap \frac{1}{M'}D\right) > \frac{1}{2}.$$

In other words, the median of the Euclidean norm with respect to μ is not larger than $\frac{1}{M'}$. Since F is αn -concave, by standard concentration inequalities for the Euclidean norm with respect to log-concave measures (it follows, e.g., from Theorem III.3 in [MS], due to Borell),

$$\mathbb{E}_\mu |x|^2 < \frac{c}{(M')^2}.$$

Combining definition (2) and the fact that $L_F^2 = \left(\frac{F(0)}{\int_K F}\right)^{\frac{2}{n}} \det(M_F)^{\frac{1}{n}}$ where M_F is the covariance matrix, we get that

$$\left(\frac{\int_K F(x)dx}{F(0)}\right)^{\frac{2}{n}} nL_F^2 \leq \mathbb{E}_\mu |x|^2 < \frac{c}{(M')^2}.$$

Since $\int_K F(x)dx \geq \text{Vol}\left(\frac{1}{M'}D \cap K\right) \geq \frac{1}{2}\text{Vol}\left(\frac{1}{M'}D\right)$ and $F(0) = 1$, we obtain that $L_F^2 < \frac{c'}{n\text{Vol}(D)^{2/n}} < \text{Const}$. \square

Proof of Theorem 1.2: We shall use the notion of l -ellipsoid, and Pisier's estimate for $M(K)M^*(K)$. We refer the reader to [P] or [MS] for definitions and proofs. Let $K \subset \mathbb{R}^n$ be a body. There exist a linear image \tilde{K} of K such that its l -ellipsoid is the standard Euclidean ball. By Pisier's estimate,

$$M^*(\tilde{K})M(\tilde{K}) < c \log d_{BM}(K, D) < c' \log n.$$

According to Corollary 3.2, there exists an αn -concave function F supported on \tilde{K} , with $\alpha = cM(\tilde{K})M^*(\tilde{K})$ and $L_F < c_1$. By Lemma 2.3 we get that $L_{K_F} < c_2$. From Lemma 2.2,

$$d_{BM}(K, K_F) \leq d_G(\tilde{K}, K_F) < c\alpha < c'M(\tilde{K})M^*(\tilde{K}) < C \log n.$$

This finishes the proof. \square

4 The quasi-convex case

With an arbitrary body $K \subset \mathbb{R}^n$ associated a special ellipsoid, called an M -ellipsoid. An M -ellipsoid may be defined by the following theorem (see [M], or chapter 7 in the book [P]):

Theorem 4.1 *Let $K \subset \mathbb{R}^n$ be a body. Then there exists an ellipsoid $\mathcal{E} \subset \mathbb{R}^n$ with $\text{Vol}(\mathcal{E}) = \text{Vol}(K)$ such that*

$$\text{Vol}(K \cap \mathcal{E})^{1/n} > c\text{Vol}(K)^{1/n}$$

where $c > 0$ is a numerical constant. We say that \mathcal{E} is an M -ellipsoid of K (with constant c).

Let $K \subset \mathbb{R}^n$ be a centrally symmetric quasi-convex body with constant C (in short “a C -quasi-body”). Assume that $\text{Vol}(K) = 1$ and that $\text{conv}(K)$ is in M -position. Moreover, we may assume that $\text{conv}(K)$ is in a 1-regular M -position, so that $\text{conv}(K) \subset \text{Vol}^{1/n}(\text{conv}(K))n^2D$ (e.g. [P] or [GM]). Let us build the following function on $\text{conv}(K)$:

$$F_K(x) = \begin{cases} 1 & |x| \leq \sqrt{n} \\ \left(1 - \frac{|x| - \sqrt{n}}{M_x - \sqrt{n}}\right)^{\alpha n} & |x| > \sqrt{n} \end{cases}$$

for some $\alpha > 0$ to be determined later, where

$$M_x = \sup \left\{ r > 0; r \frac{x}{|x|} \in \text{conv}(K) \right\}.$$

F_K is not log-concave, yet we may still consider the centrally symmetric set $K_{F_K} \subset \mathbb{R}^n$, defined in Section 2. Note that the restriction of F_K to any straight line through the origin is αn -concave on its support, hence it is possible to apply Lemma 2.2 or Lemma 2.3. We start with a one dimensional lemma.

Lemma 4.2 *Let $0 < a < b$ and $\alpha > 1$ be such that $b > 2a(1 + \frac{\alpha}{e})$. Let n be a positive integer. Then,*

$$\int_a^b \left(1 - \frac{t-a}{b-a}\right)^{\alpha n} t^n dt < \left(\frac{c_1}{\alpha}\right)^n \int_a^b t^n dt$$

where $c_1 > 0$ is a numerical constant.

Proof: Denote the integral on the left by I and the integral on the right by $J = \frac{1}{n+1} [b^{n+1} - a^{n+1}]$. Changing variables $s = \frac{t-a}{b-a}$ we get that

$$\begin{aligned} I &= (b-a) \int_0^1 (1-s)^{\alpha n} (a + (b-a)s)^n ds \\ &= (b-a) \sum_{i=0}^n \binom{n}{i} a^{n-i} (b-a)^i \int_0^1 (1-s)^{\alpha n} s^i ds \end{aligned}$$

and using (4),

$$I = (b-a) a^n \sum_{i=0}^n \frac{\binom{n}{i}}{(\alpha n + i + 1) \binom{\alpha n + i}{i}} \left(\frac{b-a}{a}\right)^i.$$

The estimate (3) together with some trivial inequalities yields that

$$I \leq \frac{b-a}{\alpha n} a^n \sum_{i=0}^n \left(\frac{e}{\alpha}\right)^i \left(\frac{b-a}{a}\right)^i = \frac{b-a}{\alpha n} a^n \frac{q^{n+1} - 1}{q - 1}$$

where $q = \frac{e(b-a)}{\alpha a}$. We assumed that $q \geq 2$, hence

$$I \leq \frac{2}{en} (aq)^{n+1} = \frac{2}{en} \left(\frac{e}{\alpha}\right)^n (b-a)^{n+1} < \left(\frac{c}{\alpha}\right)^n J.$$

□

Next we show that for a suitable value of α , which is just a numerical constant, most of the mass of F_K is not far from the origin.

Lemma 4.3 For any $\alpha > 1$,

$$\int_{\mathbb{R}^n \setminus c_2 \sqrt{n} D} F_K(x) dx < \left(\frac{c_1}{\alpha} \right)^{n-1} \text{Vol}(\text{conv}(K))$$

where c_1 is the constant from Lemma 4.2 and $c_2 = 2 \left(1 + \frac{\alpha}{\varepsilon} \right) > 1$.

Proof: Note that,

$$\int_{\mathbb{R}^n \setminus \sqrt{n} D} F_K(x) dx = \int_{S^{n-1}} \int_{\sqrt{n}}^{\max\{M_\theta, \sqrt{n}\}} \left(1 - \frac{r - \sqrt{n}}{M_\theta - \sqrt{n}} \right)^{\alpha n} r^{n-1} dr d\theta$$

where $d\theta$ is the induced surface area measure over the sphere. Let $E = \{\theta \in S^{n-1}; M_\theta > c_2 \sqrt{n}\}$. By Lemma 4.2,

$$\begin{aligned} & \int_{\mathbb{R}^n \setminus c_2 \sqrt{n} D} F_K(x) dx \\ & < \int_E \int_{\sqrt{n}}^{M_\theta} \left(1 - \frac{r - \sqrt{n}}{M_\theta - \sqrt{n}} \right)^{\alpha n} r^{n-1} dr d\theta \\ & < \left(\frac{c_1}{\alpha} \right)^{n-1} \int_E \int_{\sqrt{n}}^{M_\theta} r^{n-1} dr d\theta < \left(\frac{c_1}{\alpha} \right)^{n-1} \text{Vol}(\text{conv}(K)). \end{aligned}$$

□

Lemma 4.4 Assume that $K \subset \mathbb{R}^n$ is an A -quasi-body of volume one, and that $\text{conv}(K)$ is in a 1-regular M -position with constant B . Then for $\alpha = c_3(A, B)$,

$$L_{F_K} < c_4(A, B)$$

where $c_3(A, B), c_4(A, B)$ depend solely on their parameters, not on K or n .

Proof: Note that $v^n := \text{Vol}(\text{conv}(K)) < A^n \text{Vol}(K) = A^n$. By Brunn-Minkowski (e.g. [Sch]), the function $f(t) = \text{Vol}(\text{conv}(K) \cap t)$ is n -concave. Hence,

$$\text{Vol}(\text{conv}(K) \cap \sqrt{n} D)^{1/n} \geq \frac{1}{cv} \text{Vol}(\text{conv}(K) \cap cv \sqrt{n} D)^{1/n} > \frac{Bv}{cv} = \frac{B}{c}$$

where the constant $c > 1$ satisfies $c\sqrt{n} \text{Vol}(D)^{1/n} > 1$, so that $\text{Vol}(cv \sqrt{n} D) > \text{Vol}(\text{conv}(K))$. Let $\alpha > 2c_1 c \frac{A}{B}$. By Lemma 4.3,

$$\begin{aligned} & \int_{\mathbb{R}^n \setminus c_2 \sqrt{n} D} F_K(x) dx < \left(\frac{B}{2cA} \right)^{n-1} \text{Vol}(\text{conv}(K)) \\ & < A \left(\frac{B}{2c} \right)^{n-1} \text{Vol}(K) < \frac{2cA}{B} \frac{1}{2^{n-1}} \text{Vol}(\text{conv}(K) \cap \sqrt{n} D) \end{aligned}$$

for $c_2 = c' \frac{A}{B}$. Define a measure by $\mu(E) = \frac{\int_E F_K(x) dx}{\int_{\mathbb{R}^n} F_K(x) dx}$. Since F_K equals 1 on $\text{conv}(K) \cap \sqrt{n}D$, we get that

$$\mu(\mathbb{R}^n \setminus c_2 \sqrt{n}D) < \frac{2cA}{B} \frac{1}{2^{n-1}}.$$

Since $\text{conv}(K) \subset \text{Vol}(\text{conv}(K))^{1/n} n^2 D$,

$$\mathbb{E}_\mu |x|^2 < c_2^2 n + \frac{2cA}{B} \frac{1}{2^{n-1}} \text{Vol}(\text{conv}(K))^{1/n} n^2 < c' \frac{A^2}{B} n.$$

Therefore $L_{F_K}^2 = L_\mu^2 < c' \frac{A^2}{B} \left(\frac{F_K(0)}{\int F_K} \right)^{\frac{2}{n}}$. Note that $F_K(0) = 1$. Since $\int F_K \geq \text{Vol}(\text{conv}(K) \cap \sqrt{n}D)$, we conclude that

$$L_{F_K}^2 < c' \frac{A^2}{B} \frac{c^2}{B^2} = c_4(A, B).$$

□

Proof of Theorem 1.4: Let $K \subset \mathbb{R}^n$ be a C -quasi-body. Let \tilde{K} be a linear image of K such that $\text{Vol}(\tilde{K}) = 1$ and $\text{conv}(\tilde{K})$ is in 1-regular M -position, with a universal constant $c > 0$. Consider the function $F_{\tilde{K}}$ for $\alpha = c_3(C, c)$. By Lemma 2.2, the body $T = \tilde{K}_{F_{\tilde{K}}}$ satisfies

$$d_G(\tilde{K}, T) < c' c_3 d_G(\text{conv}(\tilde{K}), T) < c''(C)$$

for some function $c''(C) > 0$. Also, by Lemma 2.3 and Lemma 4.4,

$$L_T < \tilde{c} L_{F_{\tilde{K}}} < \bar{c}(C)$$

for some $\bar{c}(C)$ a function of C . This completes the proof. □

5 Consequences of the proof

Here we collect a few results which are byproducts of our methods. Our first two propositions enrich the family of convex bodies for which the “isomorphic slicing problem” has an affirmative answer. In this section $\text{Vol}(T)$ denotes the volume of a set $T \subset \mathbb{R}^n$ relative to its affine hull.

Lemma 5.1 *Let $K \subset \mathbb{R}^n$ be an isotropic body of volume one, and let $0 < \lambda < 1$ and $L_K < A$ for some $A > 1$. Then for any subspace E of dimension λn ,*

$$\text{Vol}(K \cap E)^{\frac{1}{n}} < c(A)$$

where $c(A)$ depends solely on A , and is independent of the body K and the dimension n .

Proof: Since $\mathbb{E}_K |x|^2 < nA^2$, the median of $|x|$ on K is smaller than $2\sqrt{n}A$. Denote $K' = K \cap 2\sqrt{n}AD$. Then $\text{Vol}(K') > \frac{1}{2}$. Also, given any subspace $E \subset \mathbb{R}^n$ of dimension λn ,

$$\text{Vol}(K' \cap E) \leq \text{Vol}(2\sqrt{n}AD \cap E) \leq \left(c \frac{A}{\sqrt{\lambda}}\right)^{\lambda n}.$$

Since K' is symmetric, $\text{Vol}(K') \leq \text{Vol}(K' \cap E) \text{Vol}(\text{Proj}_{E^\perp} K')$, where E^\perp is the orthogonal complement of E and Proj_{E^\perp} is the orthogonal projection onto E^\perp in \mathbb{R}^n . Therefore,

$$\text{Vol}(\text{Proj}_{E^\perp} K) \geq \text{Vol}(\text{Proj}_{E^\perp} K') \geq \frac{\text{Vol}(K')}{\text{Vol}(K' \cap E)} \geq \left(c \frac{\sqrt{\lambda}}{A}\right)^{\lambda n}.$$

We denote the polar body of K by $K^\circ = \{y \in \mathbb{R}^n; \forall x \in K, \langle x, y \rangle \leq 1\}$. By Santaló's inequality [Sa] and reverse Santaló [BM] (recall that projection and section are dual operations),

$$\begin{aligned} & \text{Vol}(K \cap E) \text{Vol}(\text{Proj}_{E^\perp} K) \\ & < \left(\frac{c}{\lambda n}\right)^{\lambda n} \left(\frac{c}{(1-\lambda)n}\right)^{(1-\lambda)n} \frac{1}{\text{Vol}(\text{Proj}_E K^\circ) \text{Vol}(K^\circ \cap E^\perp)} \\ & < \left(\frac{c'}{n}\right)^n \frac{1}{\text{Vol}(K^\circ)} < \left(\frac{c''}{n}\right)^n \frac{1}{\text{Vol}(D)^2} \text{Vol}(K) < \tilde{c}^n \text{Vol}(K). \end{aligned} \tag{8}$$

Hence,

$$\text{Vol}(K \cap E)^{\frac{1}{n}} < \tilde{c} \frac{\text{Vol}(K)^{\frac{1}{n}}}{\text{Vol}(\text{Proj}_{E^\perp} K)^{\frac{1}{n}}} < \tilde{c} \left(c \frac{A}{\sqrt{\lambda}}\right)^\lambda < c' A^\lambda$$

and the lemma is proven, with $c(A) = cA > cA^\lambda$. \square

The following proposition states that the isomorphic slicing conjecture holds for all projections to proportional dimension of bodies with a bounded isotropic constant.

Proposition 5.2 *Let $K \subset \mathbb{R}^n$ be a body with $L_K < A$, and let $0 < \lambda < 1$. Then for any subspace E of dimension λn , there exists a convex body $T \subset E$ such that*

$$d_{BM}(\text{Proj}_E(K), T) < c'(\lambda), \quad L_T < c(\lambda, A)$$

where Proj_E is the orthogonal projection onto E in \mathbb{R}^n , and $c'(\lambda), c(\lambda, A)$ are functions independent of K or n .

Proof: We may assume that K is of volume one and in isotropic position. For $x \in E$, define

$$f(x) = \text{Vol}(K \cap [E^\perp + x]).$$

For any $\theta_1, \theta_2 \in E$,

$$\int_E \langle x, \theta_1 \rangle \langle x, \theta_2 \rangle f(x) dx = \int_K \langle x, \theta_1 \rangle \langle x, \theta_2 \rangle dx.$$

Hence by Lemma 5.1,

$$L_f = (f(0))^{\frac{1}{\lambda n}} L_K < \text{Vol}(K \cap E^\perp)^{\frac{1}{\lambda n}} A < c(A)^{\frac{1}{\lambda}} A = c'(\lambda, A)$$

Let $T = K_f$. By Lemma 2.3 we know that $L_T < \tilde{c} L_f < c''(\lambda, A)$. Also, by Brunn-Minkowski (e.g. [Sch]) f is $(1 - \lambda)n$ -concave. By Lemma 2.2 $d_G(T, \text{Proj}_E(K)) < c \frac{1-\lambda}{\lambda}$. This finishes the proof. \square .

Our next proposition verifies the isomorphic slicing conjecture under the condition that at least a small portion of K (say, of volume $e^{-\sqrt{n}}$) is located not too far from the origin.

Proposition 5.3 *Let $K \subset \mathbb{R}^n$ be a body of volume one, such that $K \subset \beta n D$. Assume that $\text{Vol}(K \cap \gamma \sqrt{n} D) > e^{-\delta \sqrt{n}}$. Then there exists a body $T \subset \mathbb{R}^n$ such that*

$$d_{BM}(K, T) < c \left(1 + \frac{\beta \delta}{\gamma} \right), \quad L_T < c' \gamma$$

where $c, c' > 0$ are numerical constants.

Proof: If $K \subset 2\gamma \sqrt{n} D$ the proposition is trivial since $L_K < c' \gamma$. Assume the opposite, and denote $C = K \cap 2\gamma \sqrt{n} D$. Similarly to Section 3, we define

$$f(x) = \inf\{0 \leq t \leq 1; x \in (1-t)C + tK\}$$

and consider the density $F(x) = (1 - f(x))^{\alpha n}$ on K . It is enough to show that $\frac{V(K, 1; C, n-1)}{\text{Vol}(C)} < c \left(1 + \frac{\beta \delta}{\gamma} \right)$. Indeed, in that case for $\alpha = c' \left(1 + \frac{\beta \delta}{\gamma} \right)$, as in Proposition 3.1, we get that $\int_C F(x) dx > \frac{1}{2} \int_K F(x) dx$ and the same argument as in Corollary 3.2 shows that

$$L_{K_F} < c' \gamma, \quad d_G(K_F, K) < c \left(1 + \frac{\beta \delta}{\gamma} \right).$$

Let us bound $\frac{V(K, 1; C, n-1)}{\text{Vol}(C)}$. Define $f(t) = \text{Vol}(K \cap tD)$. According to our assumption, $\log f(\gamma \sqrt{n}) > -\delta \sqrt{n}$ and $\log f(2\gamma \sqrt{n}) < 0$. We conclude that there exists $\gamma \sqrt{n} < t_0 < 2\gamma \sqrt{n}$ with $(\log f(t_0))' < \frac{\delta}{\gamma}$. By Brunn-Minkowski inequality $\log f$ is concave, and $(\log f)'$ is decreasing. Therefore, for $t = 2\gamma \sqrt{n} \geq t_0$,

$$(\log f(t))' = \frac{\text{Vol}(K \cap tS^{n-1})}{\text{Vol}(K \cap tD)} < \frac{\delta}{\gamma}.$$

For $x \in \partial C$, we denote by ν_x the outer unit normal to C at x , if it is unique. Let $h_K(x) = \sup_{y \in K} \langle x, y \rangle$. Then (see [Sch]),

$$\begin{aligned} V(K, 1; C, n-1) &= \frac{1}{n} \int_{\partial C} h_K(\nu_x) dx \\ &= \frac{1}{n} \int_{K \cap tS^{n-1}} h_K(x) dx + \frac{1}{n} \int_{\partial C \setminus tS^{n-1}} h_C(\nu_x) dx \\ &\leq \frac{1}{n} \left(\frac{\delta}{\gamma} \text{Vol}(C) \right) \beta n + \text{Vol}(C) = \left(1 + \frac{\beta\delta}{\gamma} \right) \text{Vol}(C) \end{aligned}$$

where we used the fact that $h_K \leq \beta n$ and that $\text{Vol}(C) = \frac{1}{n} \int_{\partial C} h_C(\nu_x) dx$. This completes the proof. \square

For $K \subset \mathbb{R}^n$, the volume ratio of K is defined as,

$$v.r.(K) = \sup_{\mathcal{E} \subset K} \left(\frac{\text{Vol}(K)}{\text{Vol}(\mathcal{E})} \right)^{\frac{1}{n}}$$

where the supremum is over all ellipsoids contained in K . We denote

$$L_n(a) = \sup\{L_K ; K \subset \mathbb{R}^n \text{ is a body, } v.r.(K) \leq a\}.$$

In [BKM] it is proved that for any $\delta > 0$,

$$L_n < c(\delta) L_n(v(\delta))^{1+\delta}. \quad (9)$$

where $c(\delta), v(\delta) \approx e^{\frac{c}{1-\delta}}$. Next, we improve the dependence in (9).

Corollary 5.4 *There exists $c_1, c_2 > 0$, such that for all n ,*

$$L_n < c_1 L_n(c_2).$$

Proof: Our proof is a modification of the proof in [BKM]. As in that paper, let $K \subset \mathbb{R}^n$ be a body such that $L_K = L_n$ and K is of volume one and in isotropic position. As is proved in [BKM], there exists a subspace $F \subset \mathbb{R}^n$ of dimension $\lambda n = \lfloor \frac{n}{4} \rfloor$ such that

$$v.r.(Proj_E(K)) < c_1, \quad (\text{Vol}(K \cap E^\perp))^{4/n} > c_2.$$

As in the proof of Proposition 5.2 here, for $x \in E$, define

$$f(x) = \text{Vol}(K \cap [E^\perp + x]).$$

Then $d_G(K_f, Proj_E(K)) < c \frac{1-\lambda}{\lambda} < 3c$ and hence $v.r.(K_f) < 3cc_1$. Also,

$$L_f = (f(0))^{\frac{1}{\lambda n}} L_K = (\text{Vol}(K \cap E^\perp))^{4/n} L_K > c' L_K.$$

By Lemma 2.3, $L_{K_f} > \tilde{c}L_K = \tilde{c}L_n$. To conclude,

$$L_{\lambda n}(3cc_1) \geq L_{K_f} > \tilde{c}L_n$$

and by Proposition 1.3 in [BKM] $L_n > cL_{\lambda n}$. Hence for $m = \lceil \frac{n}{4} \rceil$ we have $L_m < c_1L_m(c_2)$ and the corollary is proved. \square .

Remark: Currently, there is no good proven bound for $M(K)M^*(K)$ in the non-symmetric case, and hence the central symmetry assumption of the body is crucial to the proof of Theorem 1.2. However, part of the statements in this paper may be easily generalize to non-symmetric bodies. In particular, Theorem 1.4, Corollary 2.5, Proposition 5.2, Proposition 5.3 and Corollary 5.4 also hold in the non-symmetric case.

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